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Cutoff corrections to Gaussian–Heisenberg crossover behaviour

I D Lawrie
Baker Laboratory, Cornell University, Ithaca, New York 14853, USA

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Abstract. The crossover behaviour between Gaussian and Heisenberg critical behaviour is studied in an isotropic n vector model, in which the effect of a lattice cutoff, $a = \Lambda^{-1}$, is imitated by means of $(\nabla^2\phi)^2$ and $(\nabla\nabla^2\phi)^2$ terms in the effective Hamiltonian. A field theoretic method is used to construct, to first order in $\epsilon = 4-d$, a crossover scaling function for the susceptibility containing the corrections to scaling which involve Λ directly. The leading corrections vanish, near the Heisenberg fixed point, as $[(T-T_c)^\nu/\Lambda]^2$ where ν is the corresponding correlation length exponent, and near the Gaussian fixed point as $[(T-T_c)^{1/2}/\Lambda]^2$.

1. Introduction

In earlier work (Lawrie 1976, to be referred to as I) we used renormalised perturbation theory to study the crossover between the Gaussian and Heisenberg critical behaviour in an isotropic n vector model described by the Landau–Wilson effective Hamiltonian density,

$$\mathcal{H}(x) = \frac{1}{2}(\nabla\phi_0(x))^2 + \frac{1}{2}m_0^2\phi_0^2(x) + \frac{1}{4!}g_0(\phi_0^2(x))^2. \tag{1.1}$$

Here $\phi_0(x)$ is an n component spin field, and $\phi_0^2(x)$ is an abbreviation for the expression $\Sigma_{\alpha=1}^n \phi_0^\alpha(x)\phi_0^\alpha(x)$. We are interested in calculating scaling forms for the thermodynamic functions near a critical point, and we assume, using magnetic terminology, that in this region m_0^2 is linear in temperature, while g_0 is temperature independent. The subscripts in (1.1) serve to distinguish the quantities appearing there from renormalised quantities which appear later. The critical temperature depends on the value of g_0, and we denote by $m_{0c}^2(g_0)$ the critical value of m_0^2. It is convenient to define a parameter,

$$\tau = m_0^2 - m_{0c}^2(g_0) \tag{1.2}$$

which is proportional to the reduced temperature $(T-T_c(g_0))/T_c(g_0)$.

In this paper, we report a similar calculation using the Hamiltonian

$$\mathcal{H}(x) = \frac{1}{2}\sum_{r=0}^{k}\binom{k}{r}\Lambda^{-2r}(\nabla^{r+1}\phi_0(x))^2 + \frac{1}{2}m_0^2\phi_0^2(x) + \frac{1}{4!}g_0(\phi_0^2(x))^2 \tag{1.3}$$

where $\binom{k}{r}$ is the binomial coefficient and Λ is a parameter with the dimensions of inverse length, or equivalently of mass. We shall consider explicitly only the cases $k=1$ and $k=2$. The case $k=1$ has already been studied by Bruce and Wallace (1976), and our reasons for wishing to examine it further will be explained shortly. Firstly, however, it will be useful to discuss the significance of the new parameter Λ. The unperturbed propagator in momentum space obtained from (1.3) is

$$\Delta_{\alpha\beta}(q^2)=\frac{\delta_{\alpha\beta}}{m_0^2+q^2(1+q^2/\Lambda^2)^k}. \tag{1.4}$$

This propagator appears in Feynman integrals, in which the range of integration is $0\leqslant|q|\leqslant\infty$. That is, the momentum cutoff in the original Hamiltonian has been removed. The effect of Λ, therefore, is to supress the contributions from regions of integration for which $q^2\gtrsim\Lambda^2$. In many formulations of the renormalisation group, a vital role is played by the existence of a finite Brillouin zone which, if approximated by a spherical region, restricts momentum integrations to the range $0\leqslant|q|\leqslant 1/a$, where a is a length of the order of the lattice constant. In applications to ordinary condensed matter, of course, such a cutoff is always present in the original physics. Evidently, this restriction can be reasonably approximated by using a propagator of the form (1.4) and retaining the infinite range of integration. Such a procedure has indeed been used before, but it should be noted that (1.4) does not really have the form necessary to represent a discrete lattice structure. A better form would be

$$\Delta_{\alpha\beta}(q^2)=\frac{\delta_{\alpha\beta}\theta(1-q^2a^2)}{m_0^2+(1-\cos a|q|)/a^2} \tag{1.5}$$

or, for small q^2,

$$\Delta_{\alpha\beta}(q^2)=\frac{\delta_{\alpha\beta}\theta(1-q^2a^2)}{m_0^2+q^2-\frac{1}{2}a^2q^4+\mathrm{O}(q^6)} \tag{1.6}$$

where θ is the unit step function. Near the boundary of the Brillouin zone, the q^4 term in the denominator of (1.6) has the opposite sign from that in (1.4). This makes the physical interpretation of Λ as an inverse lattice spacing somewhat doubtful. One might, however approximate the step function by

$$\theta(1-q^2a^2)\approx\frac{1}{1+\mathrm{e}^{\lambda(q^2a^2-1)}}. \tag{1.7}$$

where λ is a large constant. Expanding the denominator of (1.7) in powers of q^2, we have

$$\theta(1-q^2a^2)\approx\frac{1}{1+q^2a^2\lambda\,\mathrm{e}^{-\lambda}}. \tag{1.8}$$

This expression adds a q^4 term with the same sign as that appearing in (1.4) to the denominator of the propagator. We may therefore accept Λ as giving a measure of the size of the Brillouin zone, even though one must not expect our results to represent with any great accuracy the effect of a real lattice structure. By allowing for different values of k, we test, to some extent, the model dependence of this cutoff mechanism.

An alternative physical interpretation of our results can be obtained by adding to the nearest-neighbour interaction represented by the reciprocal of (1.5) a long-range interaction of the form

$$V(x_i,x_j)=\mathrm{e}^{-|x_i-x_j|\Lambda}, \tag{1.9}$$

which gives rise to a propagator of the form

$$\Delta_{\alpha\beta}=\frac{\delta_{\alpha\beta}\theta(1-q^2a^2)}{m_0^2+q^2+(\Lambda^{-2}-\frac{1}{2}a^2)q^4+O(q^6)}, \tag{1.10}$$

in the limit $a\to 0$ we obtain from this expression the form (1.4) with $k=1$. Furthermore, the limit $\Lambda\to 0$ may describe the approach to a multi-critical point of the Lifshitz type, associated with the onset of helical ordering (see, e.g., Hornreich *et al* 1975). We defer discussion of this possibility to future work.

With either interpretation of the physical significance of Λ, it is of interest to enquire how a finite value of this parameter affects the scaling functions. In particular, one can estimate the size of the region in which the universal scaling behaviour should be observed. Thus our first aim is to calculate crossover scaling functions which depend explicitly on Λ. Since Λ is an inverse length, one expects corrections to the universal scaling forms to vanish near the critical temperature, $\tau=0$, as τ^{ν}/Λ, where ν is the correlation length exponent. This expectation will be confirmed. As the precise physical interpretation of Λ is obscure, we do not expect our results to have any great quantitative significance, and we confine our explicit calculation to an evaluation of the susceptibility to order ϵ. (The dimensionality of space is, as usual, denoted by $d=4-\epsilon$.) The result of this calculation is summarised and discussed in § 4.

Our second object in this paper is more technical. The analysis presented in I enabled us to calculate crossover scaling functions which were correct to order ϵ^2 for all finite values of the parameters g_0 and τ in (1.1) and (1.2). However, because the analysis was performed with an infinite cutoff, or infinite Brillouin zone, these scaling functions did not have the correct analytic structure near the Heisenberg fixed point. Explicitly, simple dimensional analysis shows that the initial susceptibility, for example, can have only the form,

$$\chi=\tau^{-1}X_{\mathrm{G}}(g_0\tau^{-\epsilon/2}) \tag{1.11}$$

when calculated with the Hamiltonian (1.1). Scaling theories, and renormalisation group analysis with a finite Brillouin zone show, however, that the form (1.11) is correct only near the Gaussian fixed point. When the original momentum cutoff, a^{-1}, is finite, it is well known that there exists a value, g_0^* of g_0 such that, when g_0 is close to g_0^*, χ has the form

$$\chi=\tau^{-\gamma}X_{\mathrm{H}}((g_0-g_0^*)\tau^{-\phi_{\mathrm{u}}}) \tag{1.12}$$

where γ is the usual critical exponent, and $\phi_{\mathrm{u}}=-\omega\nu$ is the crossover exponent associated with g_0. In § 2, we summarise the analysis of I, and it will be seen (equations (2.13) and (2.28)) that the exponents γ and $\omega\nu$ appear in the renormalised theory. Thus $X_{\mathrm{G}}(z)$ has the form, for large z,

$$X_{\mathrm{G}}(z)=z^{2(\gamma-1)/\epsilon}Y(z^{-2\omega\nu/\epsilon}) \tag{1.13}$$

where Y is analytic for small values of its argument. The next-to-leading terms in τ therefore appear correctly as powers of $\tau^{\omega\nu}$, but the form (1.12) is not reproduced. The reason for this is that g_0^* is proportional to $a^{-\epsilon}$, and by working without a momentum cutoff, one is prevented from actually reaching the Heisenberg fixed point. Accordingly, we shall demonstrate in § 3 that by working with the Hamiltonian (1.3), with effective cutoff, Λ, instead of (1.1), the correct scaling form (1.12) can be obtained.

This discussion is somewhat superfluous for the case of the Gaussian–Heisenberg crossover, in view of the fact that Bruce and Wallace (1976) have obtained, to order ϵ^2, a scaling function with the correct analytic properties near both fixed points. We believe, however, that the methods of renormalised perturbation theory provide a valuable tool for the study of crossover problems in general. Thus one can obtain renormalisation group equations, such as (3.32)–(3.34), which represent an *exact* solution of the field theoretic model defined by the Hamiltonian (1.3), whereas the analogous relations obtained by other methods, such as that used by Bruce and Wallace, normally involve neglecting certain non-leading terms, including the corrections we wish to calculate. Furthermore, our equations can in principle be integrated order by order in ϵ, yielding scaling functions directly from the original theory. We thus avoid the procedure of matching two different approximations, one valid inside the critical region, and one outside, which appears to be an essential ingredient of the methods developed, for example, by Rudnick and Nelson (1976) and by Nelson and Domany (1976). It is therefore reassuring to find that the apparent limitation we have discussed can be overcome. Indeed, as we shall show in § 3, all that is necessary to obtain the correct scaling form (1.12) is to re-define the variable z in (1.11), in terms of appropriate *non-linear scaling fields* of the Gaussian fixed point. When this is done, (1.13) reproduces an expression of the correct form, (1.12), in the appropriate limit. To obtain numerical results, one may set $\epsilon = 1$, and scale the temperature so that $z = \tau^{-1/2}$. In this case, the results of I are affected only by a change of scale factor. These results are explained in a self-contained form in § 4.

2. Summary of renormalisation group structure in the non-cutoff theory

In the field theory defined by (1.1), the ostensible purpose of a renormalisation prescription is to obtain a renormalised field,

$$\phi^\alpha(x) = Z_3^{-1/2}\phi_0^\alpha(x), \tag{2.1}$$

such that the correlation functions, $\langle\phi^{\alpha_1}(x_1)\ldots\phi^{\alpha_s}(x_s)\rangle$ are free from ultraviolet divergences for $d \leq 4$, provided that they are expressed in terms of a renormalised mass and a renormalised coupling constant.

There exists a particularly useful class of renormalisation prescriptions in which, of the renormalised quantities, only the mass depends on the temperature variable, m_0^2. To obtain such a prescription, it is convenient to introduce an arbitrary parameter, μ, with the dimensions of mass, and to define the renormalised parameters u and t via

$$g_0 = \mu^\epsilon u Z_1 Z_3^{-2}, \tag{2.2}$$

$$\tau = Z_m \mu^2 t, \tag{2.3}$$

where τ is the variable defined by (1.2). The renormalisation constants, Z_1, Z_3 and Z_m, appearing in (2.1)–(2.3) are defined by evaluating certain selected correlation functions in momentum space, and specifying the values they assume at some given momentum points, which are in general functions of μ. For the class of prescriptions in which we are interested, the Z_i are functions of u alone. A particularly economical prescription belonging to this class, due to t'Hooft and Veltman (1972), was used in I. This prescription involves picking out poles as a function of dimensionality, d, which occur in the correlation functions at $\epsilon = 0$. As these poles do not appear in the cutoff theory, such a prescription will not carry over into the later parts of this work, and an alternative prescription will be necessary. The structure to be described in the remainder of this section depends only on the features we have already mentioned, which are common to both prescriptions in the non-cutoff theory.

The theory is most simply expressed in terms of the unrenormalised one-particle-irreducible correlation functions, $\Gamma_0^{(s)}$, associated with the expectation values of products of s unrenormalised fields, ϕ_0^α, and their renormalised counterparts, $\Gamma^{(s)}$. These two sets of functions are related by

$$\Gamma_0^{(s)} = Z_3^{-s/2}\Gamma^{(s)}. \tag{2.4}$$

Scaling behaviour is contained in the renormalisation group equation for the renormalised functions, namely

$$\left(t\frac{\partial}{\partial t} - \nu(u)\mu\frac{\partial}{\partial \mu} - B(u)\frac{\partial}{\partial u} + \frac{s}{2}\nu(u)\gamma_3(u)\right)\Gamma^{(s)}(p_1, \dots p_s; u, t, \mu) = 0. \tag{2.5}$$

This equation simply expresses the fact that the *unrenormalised* correlation functions are independent of the arbitrary parameter μ. It is obtained by differentiating (2.4) with respect to μ at fixed g_0 and τ. Straightforward application of the chain rule shows that the coefficients are given by

$$\beta(u) = \mu\frac{\partial u}{\partial \mu}\bigg|_{g_0\tau} = \epsilon\left[\frac{\mathrm{d}}{\mathrm{d}u}\ln\left(u\frac{Z_1}{Z_3^2}\right)\right]^{-1}, \tag{2.6}$$

$$\gamma_m(u) = \frac{1}{\mu t}\frac{\partial}{\partial \mu}(\mu^2 t)\bigg|_{g_0\tau} = \beta(u)\frac{\mathrm{d}}{\mathrm{d}u}\ln Z_m, \tag{2.7}$$

$$\gamma_3(u) = \mu\frac{\partial}{\partial \mu}\ln Z_3 = \beta(u)\frac{\mathrm{d}}{\mathrm{d}u}\ln Z_3, \tag{2.8}$$

$$\nu(u) = (2 + \gamma_m(u))^{-1}, \tag{2.9}$$

$$B(u) = \nu(u)\beta(u). \tag{2.10}$$

It is well known that $\beta(u)$ has, in perturbation theory, zeros at $u = 0$ and $u = u^*$, where u^* is of order ϵ, while $\beta(u)$ and $B(u)$ are negative for $0 < u < u^*$. The usual critical exponents are then given by

$$\nu = \nu(u^*), \tag{2.11}$$

$$\eta = \gamma_3(u^*), \tag{2.12}$$

$$\gamma = \nu(2 - \eta), \tag{2.13}$$

and corrections to scaling near the critical point are governed by the exponent

$$\omega = \beta'(u^*). \tag{2.14}$$

The thermodynamic functions in which one is ultimately interested are most closely related to the unrenormalised correlation functions, $\Gamma_0^{(s)}$. To obtain scaling functions for these, two steps are necessary. Firstly, one may obtain a solution of (2.5) in the form

$$\Gamma^{(s)}(p_i, u, t, \mu) = \Gamma^{(s)}(p_i; \bar{u}(t), 1, \bar{\mu}(t)) \exp\left(-\frac{s}{2}\int_1^t \frac{\mathrm{d}t'}{t'}\nu(\bar{u}(t'))\gamma_3(\bar{u}(t'))\right). \tag{2.15}$$

The effective mass and coupling constant, $\bar{\mu}$ and $\bar{u}$, in this expression are defined by

$$\ln t = \int_u^{\bar{u}(t)} \frac{du'}{B(u')}, \tag{2.16}$$

$$\ln\left(\frac{\bar{\mu}(t)}{\mu}\right) = \int_u^{\bar{u}(t)} \frac{\nu(u')\,du'}{B(u')}, \tag{2.17}$$

or equivalently, by the differential equations

$$t\frac{\partial \bar{u}}{\partial t} = B(\bar{u}), \tag{2.18}$$

$$t\frac{\partial}{\partial t}\bar{\mu} = \nu(\bar{u})\bar{\mu}, \tag{2.19}$$

with the initial conditions $\bar{u}(1) = u$ and $\bar{\mu}(1) = \mu$. In order to keep track of powers of ϵ, it is convenient to define

$$v = u/u^*, \tag{2.20}$$

$$\bar{v}(t) = \bar{u}(t)/u^*. \tag{2.21}$$

Equation (2.16) can then be written

$$t^{-\epsilon/2} = v^{-1}(1-v)^{\epsilon/2\omega\nu}\bar{v}(1-\bar{v})^{-\epsilon/2\omega\nu}\exp\left(\int_v^{\bar{v}} F(v')\,dv'\right). \tag{2.22}$$

We arrive at (2.22) by writing

$$\frac{1}{B(u^*v)} = \frac{2}{\epsilon u^*}\left(\frac{1}{v} + \frac{\epsilon/2\omega\nu}{1-v} + F(v)\right). \tag{2.23}$$

The function $F(v)$ defined by (2.23) is regular at the end points, $v = 0$, and $v = 1$, and in the explicit calculation of I, $F(v)$ was found to contribute to the final result only at order ϵ^3.

The second step is to express (2.15) and (2.22) in terms of unrenormalised quantities. To do this, we rewrite (2.6)–(2.8) in integral form, and obtain

$$\mu^{-\epsilon} = \frac{u^*}{g_0}v(1-v)^{-\epsilon/\omega}\exp\left(\int_0^v f(v')\,dv'\right), \tag{2.24}$$

$$Z_m^{-\epsilon/2} = (1-v)^{\epsilon/\omega[1-(1/2\nu)]}\exp\left(\int_0^v (F(v')-f(v'))\,dv'\right), \tag{2.25}$$

$$Z_3 = \exp\left(\int_0^v dv'\frac{\gamma_3(u^*v')\nu(u^*v')}{B(u^*v')/u^*}\right). \tag{2.26}$$

The exponential factor in (2.15) can be written

$$\exp\left(-\frac{s}{2}\int_v^{\bar{v}} dv'\frac{\nu(u^*v')\gamma_3(u^*v')}{B(u^*v')/u^*}\right). \tag{2.27}$$

The function $f(v)$ appearing in (2.24) and (2.25) is defined by writing for $(\beta(u^*v))^{-1}$ an equation analogous to (2.23), and is exactly cancelled in the final result, (2.29). The constant of integration, (u^*/g_0), in (2.24) is obtained by noting that in perturbation theory, (2.2) gives

$$\mu^{-\epsilon}g_0 = u^*v + O(v^2). \tag{2.28}$$

We now eliminate μ, v and t between equations (2.3) and (2.22)–(2.24) to obtain

$$\tau^{-\epsilon/2} = \frac{u^*}{g_0}\bar{v}(1-\bar{v})^{-\epsilon/2\omega\nu} \exp\left(\int_0^{\bar{v}} F(v')\,dv'\right). \tag{2.29}$$

In the same way, we may obtain $\bar{\mu}$ as a function of τ and $\bar{v}$, namely

$$\bar{\mu} = \tau^{1/2} \exp\int_0^{\bar{v}} \frac{dv'}{B(u^*v')/u^*}(\nu(u^*v') - \tfrac{1}{2}). \tag{2.30}$$

Finally, from (2.4), (2.15), (2.25) and (2.26) we obtain

$$\Gamma_0^{(s)}(p_i; g_0, \tau) = \Gamma^{(s)}(p_i; u^*\bar{v}, 1, \bar{\mu}) \exp\left(-\frac{s}{2}\int_0^{\bar{v}} \frac{\nu(u^*v')\gamma_3(u^*v')\,dv'}{B(u^*v')/u^*}\right). \tag{2.31}$$

From (2.29) we obtain $\bar{v}$ as a function of the scaling variable

$$z = g_0/u^*\tau^{\epsilon/2}, \tag{2.32}$$

and (2.30) gives the scaling form of $\Gamma_0^{(s)}$ as a function of τ and z. Each of the functions appearing on the right-hand side of (2.31) is to be calculated from renormalised perturbation theory. Provided that z is treated as being independent of ϵ, this perturbation expansion is also an expansion in powers of ϵ. When z is large, (2.28) shows that

$$\bar{v} \approx 1 - z^{-2\omega\nu/\epsilon} \approx 1 - g_0^{-2\omega\nu/\epsilon}\tau^{\omega\nu}. \tag{2.33}$$

The next-to-leading terms in this limit thus have the correct behaviour as functions of τ, but they do not contain the expected factor $(g_0 - g_0^*)$. We shall see how to correct this defect in the next section.

3. Introduction of a momentum cutoff

We now wish to extend the analysis of the last section to cover the model Hamiltonian (1.3), with $k \neq 0$. As before, we define renormalised quantities, ϕ, $\tilde{u}$ and t, via

$$\phi_0^\alpha = Z_3^{1/2}\phi^\alpha, \tag{3.1}$$

$$g_0 = \mu^\epsilon \tilde{u} Z_1 Z_3^{-2}, \tag{3.2}$$

$$m_0^2 = \mu^2 t Z_m + m_{0c}^2(g_0, \Lambda). \tag{3.3}$$

The reason for the tilde sign in (3.2) will become clear later. For $d \leqslant 4$, the only ultraviolet divergences in the theory occur as poles at $\epsilon = 0$ in self-energy insertions, and these occur only for $k = 1$. Our renormalisation prescription in this case will be designed to ensure that the renormalised correlation functions remain finite in the limit $\Lambda \to \infty$. It is no longer possible to give a prescription such that the Z_i depend on $\tilde{u}$ alone, but one can ensure that they are independent of t. We do this by specifying the values of certain correlation functions at $t = 0$. The critical mass, $m_{0c}^2(g_0, \Lambda)$ is defined by

$$\Gamma_0^{(2)}(p^2 = 0; g_0, \tau = 0, \Lambda) = 0. \tag{3.4}$$

Once again, the one-particle-irreducible correlation functions satisfy

$$\Gamma_0^{(s)} = Z_3^{-s/2}\Gamma^{(s)}, \tag{3.5}$$

and so (3.4) is equivalent to the statement

$$\Gamma^{(2)}(p^2=0; \tilde{u}, t=0, \Lambda, \mu)=0. \tag{3.6}$$

The remaining conditions necessary to define the Z_i cannot be imposed at zero momentum, owing to infrared divergences caused by the choice $t=0$. They must be imposed with momenta fixed at some finite, non-exceptional values. That is to say the momentum arguments, p_i, must be such that no non-trivial partial sum of them vanishes. A convenient set of conditions is

$$\Gamma^{(2)}(p^2=\mu^2; \tilde{u}, 0, \Lambda, \mu) = -\mu^2(1+\mu^2/\Lambda^2)^k, \tag{3.7}$$

$$\Gamma^{(4)}(p_i \,.\, p_j = \mu^2(\delta_{ij}-\tfrac{1}{4}); \tilde{u}, 0, \Lambda, \mu) = \mu^{\epsilon}\tilde{u}. \tag{3.8}$$

These two conditions serve to define Z_1 and Z_3, and are sufficient to renormalise the massless theory. The renormalisation constant, Z_m, is defined by

$$\Gamma^{(2)}_{\phi^2}(p_1^2=p_2^2=q^2=\mu^2; \tilde{u}, 0, \Lambda, \mu) = Z_3 Z_m, \tag{3.9}$$

where the correlation function $\Gamma^{(2)}_{\phi^2}$ is the two-point function $\Gamma^{(2)}$ with an insertion of the operator ϕ^2 carrying momentum q. The conditions (3.6)–(3.9) are sufficient to define renormalised correlation functions $\Gamma^{(s)}(p_i; \tilde{u}, t, \Lambda, \mu)$ which are finite in the limit $\Lambda\to\infty$. In fact, in this limit they reduce to a scheme described by Zinn-Justin (1973), apart from an unimportant change in the renormalisation point.

To one-loop order in perturbation theory, the explicit expressions we obtain for the renormalisation constants are

$$Z_1 = 1+\left(\frac{n+8}{6}\right)\left(\frac{1-\frac{1}{2}\epsilon}{\epsilon}\right)\boldsymbol{B}(1+\tfrac{1}{2}\epsilon, 1-\tfrac{1}{2}\epsilon)[\boldsymbol{B}(1-\tfrac{1}{2}\epsilon, 1-\tfrac{1}{2}\epsilon)-I_k(\mu/\Lambda)]S_d\tilde{u}, \tag{3.10}$$

$$Z_3 = 1, \tag{3.11}$$

$$Z_m = 1+\left(\frac{n+2}{6}\right)\left(\frac{1-\frac{1}{2}\epsilon}{\epsilon}\right)\boldsymbol{B}(1+\tfrac{1}{2}\epsilon, 1-\tfrac{1}{2}\epsilon)[\boldsymbol{B}(1-\tfrac{1}{2}\epsilon, 1-\tfrac{1}{2}\epsilon)-I_k(\mu/\Lambda)]S_d\tilde{u}. \tag{3.12}$$

In these equations, $\boldsymbol{B}(\alpha, \beta)$ is the Euler beta function, and for $k=1$ and $k=2$ we find

$$I_1(\theta) = \theta^{\epsilon}\int_0^1 \mathrm{d}x\,\{2[x(1-x)\theta^2+x]^{-\epsilon/2}-[x(1-x)\theta^2+1]^{-\epsilon/2}\} \tag{3.13}$$

$$\begin{aligned} I_2(\theta) = I_1(\theta)+\frac{1}{2}\epsilon\theta^{\epsilon}\int_0^1 \mathrm{d}x\,\{&2x[x(1-x)\theta^2+x]^{-(1+\frac{1}{2}\epsilon)} \\ &-2x[x(1-x)\theta^2+1]^{-(1+\frac{1}{2}\epsilon)}-(1+\tfrac{1}{2}\epsilon)x(1-x)[x(1-x)\theta^2+1]^{-(2+\frac{1}{2}\epsilon)}\}, \end{aligned} \tag{3.14}$$

The geometrical factor, $S_d = 2\pi^{d/2}/[(2\pi)^d\Gamma(d/2)]$ in (3.10) and (3.12) arises, as usual, from angular integrations. Observe that each of the expressions (3.10)–(3.12) falls naturally into two parts. The first is independent of (μ/Λ) and equal to the result obtained from the non-cutoff theory while the second vanishes, for $\epsilon>0$, in the limit $(\mu/\Lambda)\to 0$. We show in appendix 1 that this property holds for all orders of perturbation theory and for all positive values of k. For finite Λ, on the other hand, there are no poles in the limit $\epsilon\to 0$.

The necessary generalisation of the renormalisation group equation, (2.5), is obtained, as before, by differentiating (3.5) with respect to μ, at fixed g_0, τ and Λ. However, since the Z_i now depend on (μ/Λ) as well as on $\tilde{u}$, the coefficients must be found by solving the simultaneous equations which result from applying the operator

$$\mu\frac{\partial}{\partial\mu}\bigg|_{g_0\tau\Lambda}=\mu\frac{\partial}{\partial\mu}\bigg|_{\tilde{u}t\Lambda}+\tilde{\beta}\frac{\partial}{\partial\tilde{u}}\bigg|_{\mu t\Lambda}-(2+\tilde{\gamma}_m)t\frac{\partial}{\partial t}\bigg|_{\tilde{u}\mu\Lambda} \tag{3.15}$$

to (3.2) and (3.3). The definitions

$$\tilde{\beta}(\tilde{u},\mu/\Lambda)=\mu\frac{\partial\tilde{u}}{\partial\mu}\bigg|_{g_0\tau\Lambda}, \tag{3.16}$$

$$\tilde{\gamma}_m(\tilde{u},\mu/\Lambda)=-\frac{1}{\mu t}\frac{\partial}{\partial\mu}(\mu^2 t)\bigg|_{g_0\tau\Lambda}, \tag{3.17}$$

are implicit in (3.15) and we also define

$$\tilde{\gamma}_3(\tilde{u},\mu/\Lambda)=\mu\frac{\partial}{\partial\mu}\ln Z_3\bigg|_{g_0m_0}, \tag{3.18}$$

$$\tilde{\nu}(\tilde{u},\mu/\Lambda)=(2+\tilde{\gamma}_m(\tilde{u},\mu/\Lambda))^{-1}, \tag{3.19}$$

$$\tilde{B}(\tilde{u},\mu/\Lambda)=\tilde{\nu}(\tilde{u},\mu/\Lambda)\tilde{\beta}(\tilde{u},\mu/\Lambda). \tag{3.20}$$

The renormalisation group equation is then

$$\left(t\frac{\partial}{\partial t}-\tilde{\nu}\mu\frac{\partial}{\partial\mu}-\tilde{B}\frac{\partial}{\partial\tilde{u}}+\frac{s}{2}\tilde{\nu}\tilde{\gamma}_3\right)\Gamma^{(s)}=0, \tag{3.21}$$

and the coefficients are given explicitly by

$$\begin{aligned}\tilde{B}(\tilde{u},\mu/\Lambda)=-\frac{1}{2}\epsilon\tilde{u}+\left(\frac{n+8}{12}\right)\left(1-\frac{n+2}{2(n+8)}\epsilon\right)(1-\tfrac{1}{2}\epsilon)\boldsymbol{B}(1+\tfrac{1}{2}\epsilon,1-\tfrac{1}{2}\epsilon)\\ \times[\boldsymbol{B}(1-\tfrac{1}{2}\epsilon,1-\tfrac{1}{2}\epsilon)-(\mu/\Lambda)^{\epsilon}K_k(\mu/\Lambda)]S_d\tilde{u}^2+\mathrm{O}(\tilde{u}^3),\end{aligned} \tag{3.22}$$

$$\begin{aligned}\tilde{\nu}(\tilde{u},\mu/\Lambda)=\frac{1}{2}+\left(\frac{n+2}{24}\right)(1-\tfrac{1}{2}\epsilon)\boldsymbol{B}(1+\tfrac{1}{2}\epsilon,1-\tfrac{1}{2}\epsilon)\\ \times[\boldsymbol{B}(1-\tfrac{1}{2}\epsilon,1-\tfrac{1}{2}\epsilon)-(\mu/\Lambda)^{\epsilon}K_k(\mu/\Lambda)]S_d\tilde{u}+\mathrm{O}(\tilde{u}^2),\end{aligned} \tag{3.23}$$

$$\tilde{\gamma}_3(\tilde{u},\mu/\Lambda)=\mathrm{O}(\tilde{u}^2) \tag{3.24}$$

where

$$K_k(\mu/\Lambda)=-\frac{1}{\epsilon}\mu\frac{\partial}{\partial\mu}[(\mu/\Lambda)^{-\epsilon}I_k(\mu/\Lambda)]. \tag{3.25}$$

Now, to all orders in perturbation theory, the function $\tilde{B}$ consists of two parts, the first of which is equal to the coefficient obtained from the non-cutoff or $\Lambda=\infty$ theory. This follows from the form of the Z_i noted above. Since the leading term is completely independent of (μ/Λ), it is possible and convenient to make a change of variable by writing

$$\tilde{u}=u+a_2(\mu/\Lambda)u^2+\mathrm{O}(u^3), \tag{3.26}$$

or

$$u=\tilde{u}-a_2(\mu/\Lambda)\tilde{u}^2+\mathrm{O}(\tilde{u}^3), \tag{3.27}$$

where the coefficients $a_s(\mu/\Lambda)$ are chosen such that

$$\tilde{B}(\tilde{u},\mu/\Lambda)\frac{\partial u}{\partial\tilde{u}}\bigg|_{\mu/\Lambda}+\tilde{\nu}(\tilde{u},\mu/\Lambda)\mu\frac{\partial u}{\partial\mu}\bigg|_{\tilde{u}}=\tilde{B}(u,0)=B(u). \tag{3.28}$$

The renormalisation group equation (3.21) then becomes

$$\left(t\frac{\partial}{\partial t}-\nu(u,\mu/\Lambda)\mu\frac{\partial}{\partial\mu}-B(u)\frac{\partial}{\partial u}+\frac{s}{2}\nu(u,\mu/\Lambda)\gamma_3(u,\mu/\Lambda)\right)\Gamma^{(s)}(p_i;u,t,\Lambda,\mu)=0 \tag{3.29}$$

where

$$\nu(u,\mu/\Lambda)=\tilde{\nu}(\tilde{u},\mu/\Lambda), \tag{3.30}$$

$$\gamma_3(u,\mu/\Lambda)=\tilde{\gamma}_3(\tilde{u},\mu/\Lambda), \tag{3.31}$$

and $\Gamma^{(s)}(p_i;u,t,\Lambda,\mu)$ means $\Gamma^{(s)}[p_i;\tilde{u}(u,\mu/\Lambda),t,\Lambda,\mu]$, the latter function being that which appears in (3.21). The solution of (3.29) is

$$\Gamma^{(s)}(p_i;u,t,\mu,\Lambda)$$
$$=\Gamma^{(s)}(p_i;\bar{u}(t),1,\bar{\mu}(t),\Lambda)$$
$$\times\exp\left(-\frac{s}{2}\int_1^t\frac{dt'}{t'}\nu(\bar{u}(t'),\bar{\mu}(t')/\Lambda)\gamma_3(\bar{u}(t'),\bar{\mu}(t')/\Lambda)\right), \tag{3.32}$$

with the auxiliary functions $\bar{u}(t)$ and $\bar{\mu}(t)$ defined by the simultaneous differential equations

$$t\frac{\partial\bar{u}}{\partial t}=B(\bar{u}),\qquad \bar{u}(1)=u, \tag{3.33}$$

$$t\frac{\partial\bar{\mu}}{\partial t}=\bar{\mu}\nu(\bar{u},\bar{\mu}/\Lambda),\qquad \bar{\mu}(1)=\mu. \tag{3.34}$$

Although this solution looks like a natural generalisation of (2.15), it is considerably more difficult to obtain. In fact, (3.29) is actually a special case of the equation studied by Schatter and Suzuki (1975), who do not, however, supply a proof of their solution. We sketch a proof of our case in appendix 2.

Up to this point, we have followed closely the development of § 2. The next step is to convert the results (3.32)–(3.34) into expressions involving the unrenormalised parameters, g_0 and τ, instead of u, t and μ. However, in order to obtain expressions with the correct analytic structure, we must follow a somewhat different procedure. In the non-cutoff theory of § 2, the renormalisation constants, Z_i, were calculated from perturbation theory as power series in u, in which the coefficient of u^s was of order ϵ^{-s}. For this reason, one could not use equations (2.1)–(2.3) directly to recover the unrenormalised theory. Even though u is taken to be of order ϵ, each term of the ϵ expansion would collect contributions from all orders in perturbation theory. It was therefore necessary to obtain the integral forms (2.23)–(2.25) which resulted in the unsatisfactory expression (2.28). By contrast, the Z_i in (3.10)–(3.12) contain no poles, provided that (μ/Λ) is finite, and by taking u to be of order ϵ, we can solve (3.1)–(3.3) directly, order by order in ϵ. In particular, we can now find a finite value, g_0^* of g_0 such that u has its fixed point value, u^*. For this particular value of g_0, equation (3.29) predicts, as expected, a pure power law behaviour for the thermodynamic functions, as functions of temperature.

We define, once again,

$$v = u/u^*, \tag{3.35}$$

$$\bar{v} = \bar{u}/u^*. \tag{3.36}$$

Using equation (3.28) we obtain a differential equation for $a_2(\mu/\Lambda)$ whose solution is

$$a_2(\mu/\Lambda) = \frac{n+8}{6\epsilon}\left(1 - \frac{n+2}{2(n+8)}\epsilon\right)(1-\tfrac{1}{2}\epsilon)\boldsymbol{B}(1+\tfrac{1}{2}\epsilon, 1-\tfrac{1}{2}\epsilon)S_d(I_k(\mu/\Lambda) - I_k(\infty)). \tag{3.37}$$

The constant of integration, $I_k(\infty)$, is chosen so that $a_2(\mu/\Lambda)$ is regular at $\epsilon = 0$. It is difficult to eliminate explicitly the arbitrary parameter μ, and instead, we assign it a definite value,

$$\mu = \Lambda. \tag{3.38}$$

We use the last two equations, together with (3.2), (3.10), (3.11) and (3.26) to obtain

$$g_0 = \Lambda^\epsilon u^* v + O(\epsilon^3). \tag{3.39}$$

Thus, to order ϵ, we have

$$g_0^* = \Lambda^\epsilon u^* = \Lambda^\epsilon \frac{48\pi^2}{n+8}\epsilon, \tag{3.40}$$

and

$$v = g_0/g_0^*. \tag{3.41}$$

The differential equation (3.33) is identical to (2.18) and has the solution

$$t^{-\epsilon/2} = v^{-1}(1-v)^{\epsilon/2\omega\nu}\bar{v}(1-\bar{v})^{-\epsilon/2\omega\nu} \tag{3.42}$$

to this order. The exponent $\omega\nu$ has, to order ϵ, the value $\epsilon/2$, but we refrain from evaluating it, in order to make the analytic structure clear. Using

$$\tau = m_0^2 - m_{0c}^2(g_0, \Lambda) = \mu^2 t Z_m, \tag{3.43}$$

and equations (3.38)–(3.41), we replace (2.28) by

$$\tau^{-\epsilon/2} = (g_0^*/g_0)[1-(g_0/g_0^*)]^{\epsilon/2\omega\nu}(\Lambda^2 Z_m)^{-\epsilon/2}\bar{v}(1-\bar{v})^{-\epsilon/2\omega\nu}. \tag{3.44}$$

The function $F(v)$ does not contribute to order ϵ, and has been omitted from (3.44). The definition (2.31) should now be replaced by

$$z = (g_0/g_0^*)[1-(g_0/g_0^*)]^{-\epsilon/2\omega\nu}(\Lambda^2 Z_m)^{\epsilon/2}\tau^{-\epsilon/2}. \tag{3.45}$$

We now obtain $\bar{v}$ as the same function of z as that obtained in § 2. Near the Gaussian fixed point, when g_0 is small, we have again,

$$z \approx g_0/(u^*\tau^{\epsilon/2}). \tag{3.46}$$

However, near the Heisenberg fixed point, when $g_0 \approx g_0^*$, we obtain the correct form,

$$z^{-\epsilon/2\omega\nu} \propto [1-(g_0/g_0^*)]\tau^{\omega\nu}. \tag{3.47}$$

To obtain the universal parts of the scaling functions, that is, the parts which remain when (τ/Λ) is sufficiently small, and which should not depend explicitly on Λ, we observe that for small τ, (3.34) implies

$$\bar{\mu} \propto \tau^\nu, \tag{3.48}$$

where

$$\nu = \nu(u^*, 0), \tag{3.49}$$

is the usual correlation length exponent. In this region, therefore, it is sufficient to write

$$t\frac{\partial \bar{\mu}}{\partial t} = \bar{\mu}\nu(\bar{u}, 0), \tag{3.50}$$

in place of (3.34). This equation is identical to (2.19), and therefore the only alteration which should be made to the results of § 2 is to replace (2.31) by (3.45).

Lastly, since we wish to obtain the full scaling function for the susceptibility, we need the complete solution of (3.34). We can integrate (3.34) order by order in ϵ, and we obtain, to order ϵ, the transcendental equation,

$$\bar{\mu} = \tau^{1/2} Z_m^{-1/2} [1 - (g_0/g_0^*)]^{-(n+2)/2(n+8)} (1-\bar{v})^{(n+2)/2(n+8)} \times \exp\left(\frac{n+2}{2(n+8)}\bar{v}[(\bar{\mu}/\Lambda)^{-\epsilon} I_k(\bar{\mu}/\Lambda) - I_k(1)]\right). \tag{3.51}$$

In order to display the analytic structure of our solutions, we define the non-linear scaling field, τ_G, associated with the Gaussian fixed point, by

$$\tau_G = \tau Z_m^{-1} [1 - (g_0/g_0^*)]^{-(n+2)/(n+8)}, \tag{3.52}$$

where Z_m is given by (3.12) with $\mu/\Lambda = 1$. As given by (3.50), $\bar{\mu}/\Lambda$ is now a function only of (τ_G/Λ) and z, since $\bar{v}$ is the solution of (3.44). The argument of the exponential in (3.50) is of order ϵ. We can therefore solve (3.50) order by order in ϵ, to obtain

$$\bar{\mu}/\Lambda = \theta \exp\left(\frac{n+2}{2(n+8)}\bar{v}(\theta^{-\epsilon} I_k(\theta) - I_k(1))\right) \tag{3.53}$$

where

$$\theta^2 = (\tau_G/\Lambda^2)(1-\bar{v})^{(n+2)/(n+8)}. \tag{3.54}$$

Notice that, to order ϵ, (3.54) can be written as

$$\theta^2 = (\tau_G/\Lambda^2)(1-\bar{v})^{(1-2\nu)/\phi_u}, \tag{3.55}$$

where ν is the correlation length exponent for a Heisenberg-like critical point, and $\phi_u = -\frac{1}{2}\epsilon + O(\epsilon^2)$ is the crossover exponent. Near the Heisenberg fixed point, we see from (3.44) that

$$(1-\bar{v}) \propto \tau^{-\phi_u}, \tag{3.56}$$

and so

$$\theta \propto \tau^{\nu}. \tag{3.57}$$

The corrections to scaling which involve Λ therefore vanish in this region, in the expected manner.

4. Crossover scaling function for the susceptibility

The initial susceptibility is given by the two-point correlation function as

$$\chi = -(\Gamma^{(2)}(p^2 = 0))^{-1}. \tag{4.1}$$

Using the results of the last section, we can write two expressions for χ, namely

$$\chi = \tau_G^{-1} X_G(g_G \tau_G^{-\epsilon/2}, \tau_G/\Lambda^2) \tag{4.2}$$

and

$$\chi = \tau_H^{-\gamma} X_H(g_H \tau_H^{-\phi_u}, \tau_H^{2\nu}/\Lambda^2). \tag{4.3}$$

These expressions are appropriate near the Gaussian and Heisenberg fixed points respectively, and the scaling functions, X_G and X_H, are analytic for small values of their arguments. The quantities τ_G, g_G, τ_H and g_H are non-linear scaling fields, whose precise definitions will be given shortly. The exponents γ and ν are the usual susceptibility and correlation length exponents associated with the critical behaviour of a Heisenberg system, given to order ϵ by

$$\gamma = 2\nu = 1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2), \tag{4.4}$$

while

$$\phi_u = -\tfrac{1}{2}\epsilon + O(\epsilon^2) \tag{4.5}$$

is the Heisenberg crossover exponent.

In the critical region, where corrections to scaling involving Λ are negligible, the scaling functions depend only on the scaling variable, z, which is defined in terms of the original parameters, g_0, τ and Λ, entering (1.1)–(1.4), by

$$z = (\Lambda^\epsilon g_0/g_0^*)[1-(g_0/g_0^*)]^{\epsilon/2\phi_u}(\tau/Z_m)^{-\epsilon/2}. \tag{4.6}$$

The fixed-point value, g_0^*, of g_0 is found to be

$$g_0^* = \Lambda^\epsilon \frac{48\pi^2}{n+8}\epsilon + O(\epsilon^2), \tag{4.7}$$

To this order, these solutions are both equivalent to the single expression

$$\bar{v} = z/(1+z) \tag{4.13}$$

and we shall make use of this fact to facilitate numerical calculations. In general, however, it should be understood that (4.11) and (4.12) are solutions of (4.9) obtained in two distinct approximations.

Our choice of non-linear scaling fields is governed so far by the condition (4.10), together with the natural requirement that τ_G and τ_H should be proportional to τ. These conditions do not, however, define them uniquely, and we consider next the corrections to scaling involving Λ. These occur in the combination

$$\theta^2 = (\tau_G/\Lambda^2)(1-\bar{v})^{(1-2\nu)/\phi_u}. \tag{4.14}$$

In writing (4.14) we have made the specific choice

$$\tau_G = (\tau/Z_m)[1-(g_0/g_0^*)]^{(2\nu-1)/\phi_u}, \tag{4.15}$$

which also implies

$$g_G = (\Lambda^\epsilon g_0/g_0^*)[1-(g_0/g_0^*)]^{\epsilon\nu/\phi_u}. \tag{4.16}$$

Using (4.11), we can write (4.14) in the form

$$\theta^2 = (\tau_G/\Lambda^2)(1+g_G\tau_G^{-\epsilon/2})^{2(1-2\nu)/\epsilon}, \tag{4.17}$$

which is appropriate near the Gaussian fixed point. When τ is small, θ^2 varies as $\tau^{2\nu}$, and by an appropriate choice of scaling fields, we can ensure that, near the Heisenberg fixed point, θ^2 has the correct scaling form

$$\theta^2 = (\tau_H^{2\nu}/\Lambda^2)(1+g_H\tau_H^{-\phi_u})^{(2\nu-1)/\phi_u}. \tag{4.18}$$

The appropriate definitions are

$$\tau_H = (\tau/Z_m)(\Lambda^\epsilon g_0/g_0^*)^{(1-2\nu)/\epsilon\nu}, \tag{4.19}$$

and

$$g_H = [1-(g_0/g_0^*)](\Lambda^\epsilon g_0/g_0^*)^{\phi_u/\epsilon\nu}. \tag{4.20}$$

With these definitions, we see that near the Gaussian and Heisenberg fixed points respectively, θ is given approximately by

$$\theta \approx \tau_G^{1/2}/\Lambda, \qquad \text{Gaussian}, \tag{4.21}$$

or

$$\theta \approx \tau_H^{\nu}/\Lambda, \qquad \text{Heisenberg}. \tag{4.22}$$

Thus the corrections to scaling associated directly with Λ vanish in the expected manner.

The susceptibility is obtained by explicit evaluation of (3.22) which, to order ϵ, gives simply

$$\chi = \bar{\mu}^{-2}. \tag{4.23}$$

Our result, in parametric form, is

$$\chi = (\Lambda^2\theta^2)^{-1} \exp\left(-\frac{n+2}{n+8}\,\bar{v}(\theta^{-\epsilon}I_k(\theta) - I_k(1))\right), \tag{4.24}$$

where the function $I_k(\theta)$ is defined, for $k=1$ and $k=2$, by (3.13) and (3.14). The scaling functions in (4.2) and (4.3) are obtained explicitly by substituting in (4.24) the appropriate approximate solutions for $\bar{v}$ and θ. In particular, it is instructive to calculate the leading terms in X_{H}, which are

$$X_{\mathrm{H}}(x, y)=1+\frac{n+2}{n+8}x+\frac{5}{12}\frac{n+2}{n+8}\epsilon y-3\frac{n+2}{(n+8)^2}x^2-\frac{5}{12}\frac{n+2}{n+8}\epsilon xy-\frac{3}{20}\frac{n+2}{n+8}\epsilon y^2+\dots. \tag{4.25}$$

Our results may be conveniently displayed in graphical form by calculating the effective exponent,

$$\gamma_{\mathrm{eff}}=-\tau\frac{\partial\ln\chi}{\partial\tau}. \tag{4.26}$$

In parametric form, we have

$$\gamma_{\mathrm{eff}}(\bar{v},\theta)=1+\frac{n+2}{2(n+8)}\epsilon\bar{v}(1-K_k(\theta)), \tag{4.27}$$

where the function $K_k(\theta)$ is defined by

$$K_k(\theta)=-\frac{1}{\epsilon}\theta\frac{\partial}{\partial\theta}(\theta^{-\epsilon}I_k(\theta)). \tag{4.28}$$

In order to express our results in terms of dimensionless quantities it is necessary to introduce a mass parameter, Λ_0, the reciprocal of which provides a basic unit of length. The dimensionless variables are

$$\Lambda'=\Lambda/\Lambda_0, \tag{4.29}$$

$$g'_{\mathrm{G}}=\Lambda_0^{-\epsilon}g_{\mathrm{G}}, \tag{4.30}$$

$$\tau'_{\mathrm{G}}=\Lambda_0^{-2}\tau_{\mathrm{G}}, \tag{4.31}$$

$$g'_{\mathrm{H}}=\Lambda_0^{-\phi_u/\nu}g_{\mathrm{H}}, \tag{4.32}$$

$$\tau'_{\mathrm{H}}=\Lambda_0^{-1/\nu}\tau_{\mathrm{H}}. \tag{4.33}$$

The dimensionless functions, $\bar{v}$ and θ are, of course, independent of Λ_0 after the substitutions (4.29)–(4.33).

In figure 1, we show γ_{eff} as a function of τ'_{H}, for various values of Λ', with $g'_{\mathrm{H}}=0$. We have set $\epsilon=1$, $n=3$, and $k=1$ in (4.27). For $\Lambda'=\infty$, we see pure power law behaviour, with $\gamma_{\mathrm{eff}}\equiv\gamma$. For finite values of Λ', however, the critical region, in which $\gamma_{\mathrm{eff}}\approx\gamma$, has a finite size which decreases as Λ' decreases, that is, as the Λ'-dependent corrections in (4.24) becomes more important.

When $g_0=0$, we find that γ_{eff} has its mean-field value, $\gamma_{\mathrm{eff}}=1$, independent of τ and Λ. Intermediate situations are shown in figures 2 and 3. Figure 2 shows γ_{eff} as a function of τ'_{H}, with $g'_{\mathrm{H}}=0{\cdot}25$ while in figure 3 we have set $g'_{\mathrm{G}}=0{\cdot}25$, and plotted γ_{eff} as a function of the scaling field, τ'_{G}. For $\Lambda'=\infty$, we see a single profile, which moves along the temperature axis as g_0 is varied. For finite values of Λ', however, the profile also changes shape as it moves. This change of shape is small near the Gaussian fixed point (figure 3), but becomes more pronounced as the Heisenberg fixed point is approached.

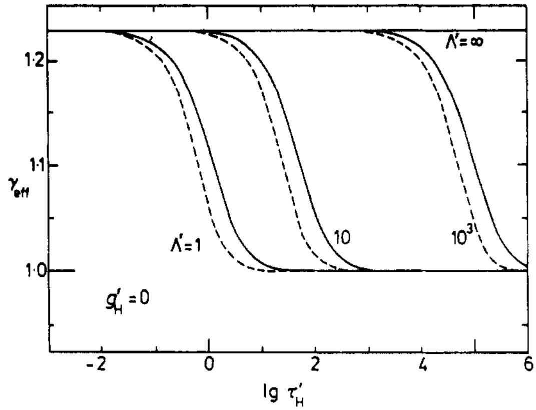

Figure 1. Plot of the effective exponent, γ_{eff}, as a function of τ'_H for various values of Λ', with $g'_H = 0$, $\epsilon = 1$ and $n = 3$. Full curves are for $k = 1$ and broken curves for $k = 2$.

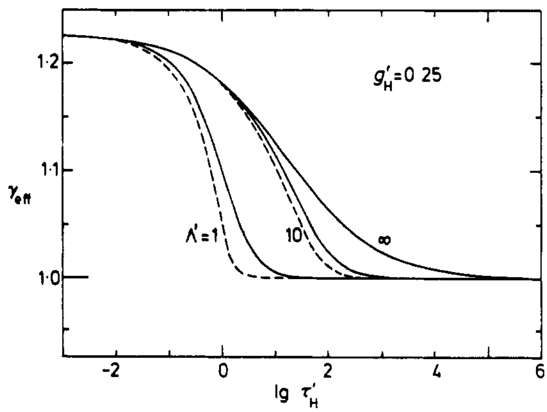

Figure 2. Plot of γ_{eff} as a function of τ'_H for $g'_H = 0{\cdot}25$ for various values of Λ'. Full curves are for $k = 1$ and broken curves for $k = 2$.

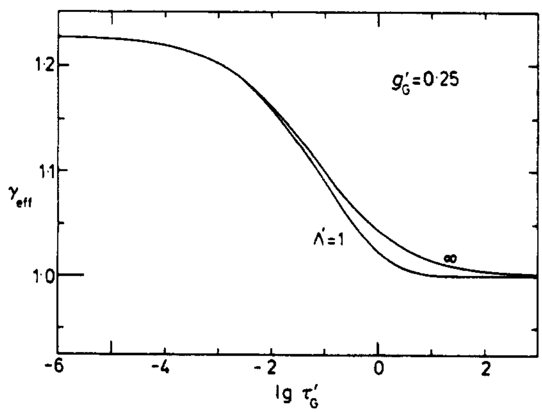

Figure 3. Plot of γ_{eff} as a function of τ'_G for $g'_G = 0{\cdot}25$ and $k = 1$, and for the indicated values of Λ'.

The broken curves in figures 1 and 2 are obtained by setting $k=2$. They suggest that the form of the cutoff corrections is relatively insensitive to the details of the mechanism by which Λ is introduced, at least within the class of models considered.

When $y=0$, we obtain, to order ϵ, the results given previously by Rudnick and Nelson (1976) and by Bruce and Wallace (1976). We may also note that there is, as is commonly expected, an eventual crossover to mean field behaviour at large values of τ, for all values of g_0 and Λ except $g_0=g_0^*$ and $\Lambda=\infty$. However, this occurs in a region far from criticality, where many other irrelevant variables, which could be ignored in the critical region, should also be important. Such behaviour is a property of our simplified model Hamiltonian, rather than of any real physical system.

Finally, we discuss one further feature of our calculation which may have some significance. The functions $I(\theta)$ and $K(\theta)$ involve exponents which depend on ϵ. As they appear in terms which are already of order ϵ, it is formally correct, in an $O(\epsilon)$ calculation, to set $\epsilon=0$ in these exponents and we have done this in order to arrive at (4.24), and (4.27). An alternative procedure would have been to retain the full values of the exponents. In this case, one would have non-analytic corrections in (4.24), of the form, $y^{1+\epsilon/2}$, etc. These exponents are indicative of a formal defect in our calculation. The operator $(\nabla^2\phi^2)$ which we have used to imitate the effect of a lattice cutoff is not, in general, an eigen-operator of the renormalisation group near the Heisenberg fixed point. A fully complete calculation of the corrections involving this operator should thus take account of mixing with other irrelevant perturbations which also have canonical dimension six in four dimensions. Since these include, for example, the non-renormalisable operator ϕ^6, it would not then be possible, using the direct methods of renormalised perturbation theory, to obtain a scaling function in closed form. We know of no calculations of this mixing, although the problem for operators of dimension four has been discussed by Brézin *et al* (1974a, b). Indeed, it is not at all clear to what extent our results are misleading in this respect. It seems reasonable to assume that any new scaling exponent associated with the gradient interaction would be, like the exponent η, associated with the operator $(\nabla\phi)^2$, at least of order ϵ^2; in that case our order ϵ calculation would be unaffected. In any event, we are not concerned here with general treatment of such perturbations, but rather with estimating the effect of a lattice cutoff which, as discussed earlier, is, at best, only approximately represented by the Hamiltonian (1.3). For this problem it seems natural to expect a scaling function $X_{\rm H}(x,y)$ as defined in (4.3) which is analytic for small values of its arguments. Our results to $O(\epsilon)$ are quite consistent with this expectation, as explained.

It is also possible that these exponents are indicative of the presence of another fixed point, namely, the isotropic Lifshitz point, associated with the limit $\Lambda\to 0$. Unfortunately, the crossover dimension associated with this point is $d=8$, and it is not possible to make any reliable statements about such behaviour on the basis of the present calculation. If the full exponents are retained, (4.27) reads

$$\gamma_{\rm eff}(\bar v,\theta)=1+\frac{n+2}{2(n+8)}\epsilon\bar v(1-\theta^{\epsilon}K_k(\theta)), \tag{4.29}$$

where the ϵ-dependent exponents in $K_k(\theta)$ are now understood to have their full values. In figure 4, we illustrate the difference between (4.29) and (4.27) by plotting $\gamma_{\rm eff}$ as a function of $\tau'_{\rm H}$ with $g'_{\rm H}=0{\cdot}25$ and $k=1$. The curves with $\Lambda'=10$ and $\Lambda'=\infty$ may be compared with those of figure 2. As one would expect, these two figures are in substantial agreement when $\tau'_{\rm H}$ is not too large, and Λ' is not too small, i.e., for small θ.

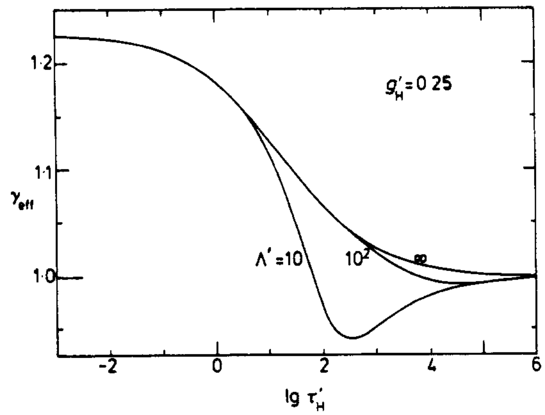

Figure 4. Plot of γ_{eff} as a function of τ'_H for $g'_H = 0{\cdot}25$, $k = 1$, and various values of Λ', retaining the full values of some ambiguous exponents discussed in the text.

The dip in figure 4 is reminiscent of similar behaviour found, for example, in the cases of Heisenberg-dipolar crossover (Bruce *et al* 1976) and of anisotropic Heisenberg systems (Nelson and Domany 1976, Bruce and Wallace 1976). However, it should probably not be taken too seriously. Indeed, one could regard the agreement of figures 2 and 4 as indicating the extent of the region over which the ϵ expansion can be expected to give sensible results, at least in leading order.

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Appendix 1

We obtain here some properties of the Feynman integrals referred to in § 3. To carry out the renormalisation, we need consider only the massless theory ($t = 0$). A general diagram with L loops and I internal lines then has the form

$$G = \prod_{i=1}^{L} \int d^d q_i \prod_{j=1}^{I} (k_j^2)^{-1}(\Lambda^{-2}k_j^2 + 1)^{-k}, \tag{A.1}$$

where the k_j are linear combinations of the loop momenta, q_i, and the external momenta p_s. Using the parametrisation due to Schwinger, we may rewrite (A.1) as

$$G = \prod_{i=1}^{L} \int d^d q_i \left(\prod_{j=1}^{I} \prod_{l=1}^{k} \int_0^\infty d\alpha_j \int_0^\infty d\beta_{jl} \right) \exp\left(-\sum_j \alpha_j k_j^2 - \sum_{jl} \beta_{jl}(\Lambda^{-2}k_j^2 + 1) \right). \tag{A.2}$$

On making the transformation, $\alpha_j \to \alpha_j - \Lambda^{-2}\Sigma_l \beta_{jl}$, this becomes

$$G = \prod_{i=1}^{L} \int d^d q_i \left(\prod_{j=1}^{I} \prod_{l=1}^{k} \int_0^\infty d\alpha_j \int_D d\beta_{jl} \right) \exp\left(-\sum_j \alpha k_j^2 - \sum_{jl} \beta_{jl} \right), \tag{A.3}$$

where D is the region $\beta_{jl} \geq 0$, $\Sigma_l \beta_{jl} \leq \Lambda^2 \alpha_j$. The integrals over the β_{jl} and the loop momenta can be performed explicitly to give an expression of the form

$$G = \mu^{dL-2I} \left(\prod_{j=1}^{I} \int_0^\infty d\alpha_j \right) \tilde{G}(\alpha_j) \prod_j \left(1 - e^{-\alpha_j \Lambda^2/\mu^2} \sum_{l=1}^{k-1} \frac{(\alpha_j \Lambda^2/\mu^2)}{l!} \right). \tag{A.4}$$

In this equation, the external momenta have been evaluated at a renormalisation point with $|p_s| \propto \mu$, and the α_j have been scaled by a factor of μ^{-2}.

The overall factor of μ^{dL-2I} is removed when calculating the dimensionless Z_i, and the properties required in § 3 are contained in the final product of (A.4). The first term in the expansion of this product is unity, and reproduces the value of the corresponding diagram in the non-cutoff theory. In the limit $\mu/\Lambda \to 0$, only this term survives, but for finite μ/Λ, the remaining terms also contribute, and poles at $\epsilon = 0$ cancel, except in certain self-energy diagrams for $k = 1$.

Appendix 2

We wish to verify that the equation

$$(\Delta + \gamma(u, \mu))\Gamma = 0, \tag{A.5}$$

where Δ is the differential operator

$$\Delta = t\frac{\partial}{\partial t} - B(u)\frac{\partial}{\partial u} - \nu(u, \mu)\frac{\partial}{\partial \ln \mu}, \tag{A.6}$$

has the solution

$$\Gamma(u, t, \mu) = \Gamma(\bar{u}(t), 1, \bar{\mu}(t)) \exp\left(-\int_1^t \frac{dt'}{t'} \gamma(\bar{u}(t'), \bar{\mu}(t')) \right), \tag{A.7}$$

where the auxiliary functions $\bar{u}$ and $\bar{\mu}$ are defined by

$$\ln t = \int_u^{\bar{u}(t)} \frac{du'}{B(u')}, \tag{A.8}$$

and

$$\ln(\bar{\mu}(t)/\mu) = \int_1^t \frac{dt'}{t'} \nu(\bar{u}(t'), \bar{\mu}(t')). \tag{A.9}$$

We may use (A.8) and (A.9) to rewrite (A.6) as

$$\Delta = t\frac{\partial}{\partial t}\bigg|_{\bar{u}\bar{\mu}} + (\nu(\bar{u}, \bar{\mu}) - D(u, \mu, t))\frac{\partial}{\partial \ln \bar{\mu}}\bigg|_{\bar{u}t}, \tag{A.10}$$

where

$$D(u, \mu, t) = \left(B(u)\frac{\partial}{\partial u} + \nu(u, \mu)\frac{\partial}{\partial \ln \mu} \right) \ln \bar{\mu}(t). \tag{A.11}$$

Further application of (A.8) and (A.9) shows that

$$D(u, \mu, t) = \nu(u, \mu) + \int_1^t \frac{dt'}{t'} \left(B(\bar{u}) \frac{\partial}{\partial \bar{u}} + D(u, \mu, t') \frac{\partial}{\partial \ln \bar{\mu}} \right) \nu(\bar{u}, \bar{\mu}). \tag{A.12}$$

Also, the chain rule gives

$$t \frac{\partial}{\partial t} \nu(\bar{u}(t), \bar{\mu}(t)) = \left(B(\bar{u}) \frac{\partial}{\partial \bar{u}} + \nu(\bar{u}, \bar{\mu}) \frac{\partial}{\partial \ln \bar{\mu}} \right) \nu(\bar{u}, \bar{\mu}), \tag{A.13}$$

or, in integral form,

$$\nu(\bar{u}(t), \bar{\mu}(t)) - \nu(u, \mu) = \int_1^t \frac{dt'}{t'} \left(B(\bar{u}) \frac{\partial}{\partial \bar{u}} + \nu(\bar{u}, \bar{\mu}) \frac{\partial}{\partial \ln \bar{\mu}} \right) \nu(\bar{u}, \bar{\mu}). \tag{A.14}$$

Subtracting (A.14) from (A.12), we have

$$D(u, \mu, t) - \nu(\bar{u}(t), \bar{\mu}(t)) = \int_1^t \frac{dt'}{t'} [D(u, \mu, t) - \nu(\bar{u}(t'), \bar{\mu}(t'))] \frac{\partial \nu(\bar{u}, \bar{\mu})}{\partial \ln \bar{\mu}}, \tag{A.15}$$

which has the solution

$$D(u, \mu, t) - \nu(\bar{u}(t), \bar{\mu}(t)) = 0. \tag{A.16}$$

This solution is unique if $D(u, \mu, t) - \nu(\bar{u}, \bar{\mu})$ and $\partial \nu(\bar{u}, \bar{\mu})/\partial \ln \bar{\mu}$ are differentiable to all orders, which we assume to be the case, at least for $t > 0$.

The original equation (A.5), therefore reduces to

$$\left(t \frac{\partial}{\partial t}\bigg|_{\bar{u}, \bar{\mu}} + \gamma(u, \mu) \right) \Gamma = 0. \tag{A.17}$$

Now we have

$$\begin{aligned} t \frac{\partial}{\partial t}\bigg|_{\bar{u}\bar{\mu}} & \int_1^t \frac{dt'}{t'} \gamma(\bar{u}(t'), \bar{\mu}(t')), \\ &= \Delta \int_1^t \frac{dt'}{t'} \gamma(\bar{u}, \bar{\mu}), \\ &= \gamma(\bar{u}(t), \bar{\mu}(t)) - \int_1^t \frac{dt'}{t'} \left(B(u) \frac{\partial}{\partial u} + \nu(u, \mu) \frac{\partial}{\partial \ln \mu} \right) \gamma(\bar{u}, \bar{\mu}), \\ &= \gamma(\bar{u}(t), \bar{\mu}(t)) - \int_1^t \frac{dt'}{t'} \left(t' \frac{\partial}{\partial t'}\bigg|_{u\mu} - t' \frac{\partial}{\partial t'}\bigg|_{\bar{u}\bar{\mu}} \right) \gamma(\bar{u}, \bar{\mu}), \\ &= \gamma(u, \mu). \end{aligned} \tag{A.18}$$

From this it follows that (A.7) is a solution of (A.17) and therefore also of (A.5).

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